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FREE CONVECTION WITH BLOWING AND SUCTION

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Abstract—The effects of uniform blowing and suction on the free convection boundary layer on a vertical plate are considered. A numerical solution of the full boundary layer equations is obtained in both cases. In the case of suction the asymptotic solution is found to be a boundary layer of constant thickness. The approach to this solution is also discussed. When fluid is blown through the plate, it is found that, at large distances from the leading edge, the boundary layer has an inner inviscid region made up of fluid that has been blown through the plate, and an outer viscous region where the fluid attains the ambient conditions.

NOMENCLATURE

- x, co-ordinate measuring distance along plate;
- y, co-ordinate measuring distance normal to plate;
- u, velocity component in the x-direction;
- v, velocity component in the y-direction;
- V, transpiration velocity;
- g, acceleration of gravity;
- T, temperature of fluid;
- T_0, temperature of ambient fluid;
- T_1, plate temperature;
- ΔT, $T_1 - T_0$;
- θ, non-dimensional temperature = $T - T_0/\Delta T$;
- ψ, stream function;
- β, coefficient of thermal expansion;
- ν, kinematic viscosity;
- κ, thermometric conductivity;
- σ, Prandtl number = ν/κ;
- τ_ω, non-dimensional skin friction;
- Q, non-dimensional heat transfer.

1. INTRODUCTION

THE PROBLEM considered in this paper is that of the effects of blowing and suction on the free convection boundary layer on a semi-infinite vertical flat plate. We consider the cases of uniform blowing and suction with the plate held at a constant temperature T_1 greater than the temperature T_0 of the ambient fluid. The boundary layer forms on the plate as a result of the buoyancy forces caused by the applied temperature difference, the effects of blowing (or suction) increase as distance from the leading edge increases.

Eichhorn [1] has considered the power law variations of plate temperature and transpiration velocity for which a similarity solution is possible. He quotes results for a uniform wall temperature and a Prandtl number of 0·73. Sparrow and Cess [2] have looked at the problem of constant plate temperature and transpiration velocity. They expanded the velocity and temperature in series in x . The leading term is the free convection solution, given by Ostrach [3], and they give the second term in the series for a Prandtl number of 0·72. This method has the disadvantage of giving accurate results only for small values of x. The accuracy could be improved by the straightforward, but laborious, procedure of computing more terms in the series. At each stage there is a fifth-order set of linear differential equations to be solved numerically.

The method of solution adopted in this paper is to obtain a step-by-step numerical solution of the boundary layer equations. The solution starts from the leading edge ($x = 0$), where the velocity and temperature profiles are given by the free convection solution, and proceeds up the plate until, in each case, the asymptotic solution is attained to the required accuracy. An overall accuracy of 3 figures was achieved and results are given for a Prandtl number of 1. The same method can be used to achieve a higher accuracy and to give results for other values of the Prandtl number.

In the case when fluid is sucked through the plate, the asymptotic solution (i.e. as $x \to \infty$) is that of a boundary layer of constant thickness. This is analogous to flow of a uniform stream over a flat plate with constant suction where Griffiths and Meredith [4] found that the asymptotic velocity profile was independent of x. Stewartson [5] showed that the approach to this asymptotic suction profile was through an essential singularity at $x = \infty$; an essential singularity at $x = \infty$ is also found in this case.

When fluid is blown through the plate, there is, for large x, a region next to the plate where temperature of the fluid is T_1 and where viscosity can be neglected. This region is formed by the fluid which has been blown through the plate. By neglecting viscosity in the Von Mises form of the boundary layer equations Aroesty and Cole [6] showed that the thickness of this inviscid region is proportional to $x^{\frac{1}{3}}$, and that the streamline which determines the outer edge of the region is the one that emerged from the plate at $x = 0$. Since they neglected viscosity, there was a discontinuity in velocity and temperature at this dividing streamline. Viscous effects are important near this discontinuity. An asymptotic expansion is obtained, centred about this dividing streamline which merges with the inviscid solution near the plate and satisfies the conditions that the fluid is at rest and has temperature T_0 well away from the plate.

2. EQUATIONS OF MOTION

The plate is fixed in a vertical position with the leading edge horizontal. x measures distance along the plate; $x = 0$ being the leading edge, and y measures distance normally outwards. u and v are the velocity components in the x and y directions respectively. The plate is held at a constant temperature T_1 greater than the ambient temperature T_0. If we make the assumption that $\Delta T/T_0 \ll 1$ then viscous dissipation can be neglected and changes in density are important only in producing buoyancy forces. The boundary layer equations are then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T - T_0) + \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2}. \tag{3}$$

The boundary conditions are

$u = 0$, $T = T_1$, $v = V$ (for blowing) or $v = -V$ (for suction) on $y = 0$, $u \to 0$, $T \to T_0$ as $y \to \infty$,

and

$$u = 0,\ T = T_0 \quad \text{at} \quad x = 0\ (y \neq 0).$$

3. NUMERICAL SOLUTION

Near the leading edge the flow is caused mainly by the buoyancy forces which arise from the difference in temperature between the plate and the ambient fluid. This suggests the following transformation of the boundary layer equations.

$$\psi = \frac{\nu^2 g\beta\Delta T}{V^3}\xi^3[f(\eta, \xi) \pm \tfrac{1}{4}\xi]$$

$$T - T_0 = \Delta T\theta(\eta, \xi)$$

where ψ is the stream function,

$$\eta = \frac{V}{\nu}\frac{y}{\xi}$$

and

$$\xi = V\left(\frac{4x}{\nu^2 g\beta\Delta T}\right)^{\frac{1}{4}}$$

The upper sign is taken throughout for suction and the lower sign for blowing. Equations (1)–(3)

become

$$\frac{\partial^3 f}{\partial \eta^3} + \theta + 3f\frac{\partial^2 f}{\partial \eta^2} - 2\left(\frac{\partial f}{\partial \eta}\right)^2 = \xi\left(\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta \partial \xi} - \frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2} \mp \frac{\partial^2 f}{\partial \eta^2}\right) \quad (4)$$

$$\frac{1}{\sigma}\frac{\partial^2 \theta}{\partial \eta^2} + 3f\frac{\partial \theta}{\partial \eta} = \xi\left(\frac{\partial \theta}{\partial \xi}\frac{\partial f}{\partial \eta} - \frac{\partial \theta}{\partial \eta}\frac{\partial f}{\partial \xi} \mp \frac{\partial \theta}{\partial \eta}\right) \quad (5)$$

where σ is the Prandtl number. The boundary conditions are

$$\theta = 1, \quad f = \frac{\partial f}{\partial \eta} = 0 \quad \text{on} \quad \eta = 0$$

and

$$\theta \to 0, \quad \frac{\partial f}{\partial \eta} \to 0 \quad \text{as} \quad \eta \to \infty.$$

Sparrow and Cess [2] expanded f and θ in the form

$$f(\eta, \xi) = f_0(\eta) \pm \xi f_1(\eta) + \dots$$
$$\theta(\eta, \xi) = \theta_0(\eta) \pm \xi\theta_1(\eta) + \dots$$

f_0 and θ_0 are given by the free convection solution of [3], and the other terms in the series are found by solving linear differential equations. This method of solution is not followed in this paper. Equations (4) and (5) are solved numerically for both the cases of suction and blowing. The numerical solution starts at the leading edge ($\xi = 0$), and proceeds step-by-step up the plate until the asymptotic velocity and temperature profiles are attained to the required accuracy.

The method is similar to that used by Merkin [7]; the idea is, knowing velocity and temperature profiles at ξ_1, to calculate them at ξ_2 where $\xi_2 > \xi_1$. The equations are first written in terms of $\partial f/\partial \eta$, then derivatives in the ξ-direction are replaced by differences and all other quantities by averages. This leads to a fifth-order set of non-linear differential equations to solve for $S(\eta)$ and $t(\eta)$ where

$$S = \left(\frac{\partial f}{\partial \eta}\right)_1 + \left(\frac{\partial f}{\partial \eta}\right)_2$$

and $t = \theta_1 + \theta_2$ (suffices 1 and 2 denoting values at ξ_1 and ξ_2 respectively). To solve these equations differences are introduced in the η-direction. This gives the two sets of non-linear algebraic equations

$$S_{j+1} - 2S_{j-1} + S_{j-1} \pm \frac{h}{4}(\xi_1 + \xi_2) \times (S_{j+1} - S_{j-1}) + \frac{h^2}{4}(S_{j+1} - S_{j-1}) \times [(3 + \lambda)(S_1 + \dots + \tfrac{1}{2}S_j) - 2\lambda\delta_j] + h^2\left[t_j - S_j^2 - \frac{\lambda}{2}S_j(S_j - 2q_{ij})\right] = 0 \quad (6)$$

$$\frac{1}{\sigma}(t_{j+1} - 2t_j + t_{j-1}) \pm \frac{h}{4}(\xi_1 + \xi_2) \times (t_{j+1} - t_{j-1}) + \frac{h^2}{4}(t_{j+1} - t_{j-1}) \times [(3 + \lambda)(S_1 + \dots + \tfrac{1}{2}S_j) - 2\lambda\delta_j] = 0 \quad (7)$$

where h is the step length in the η-direction,

$$\lambda = \frac{\xi_2 + \xi_1}{\xi_2 - \xi_1},$$

$$\delta_j = (q_1 + q_2 + \dots + \tfrac{1}{2}q_j)$$

and

$$j = 1, 2, \dots, N.$$

N must be chosen sufficiently large so that the boundary condition that S and $t \to 0$ as $\eta \to \infty$ can be replaced by $S_{N+1} = t_{N+1} = 0$. Equations (6) and (7) are solved by iteration in the following way. Taking the values S_j and t_j calculated from the previous step as a first approximation, a better approximation for the S_j is found by solving equation (6) by Newton's method. Using these S_j, equation (7) is then solved for t_j. These values of the S_j and t_j are then used as the next approximation and the process repeated until the difference between

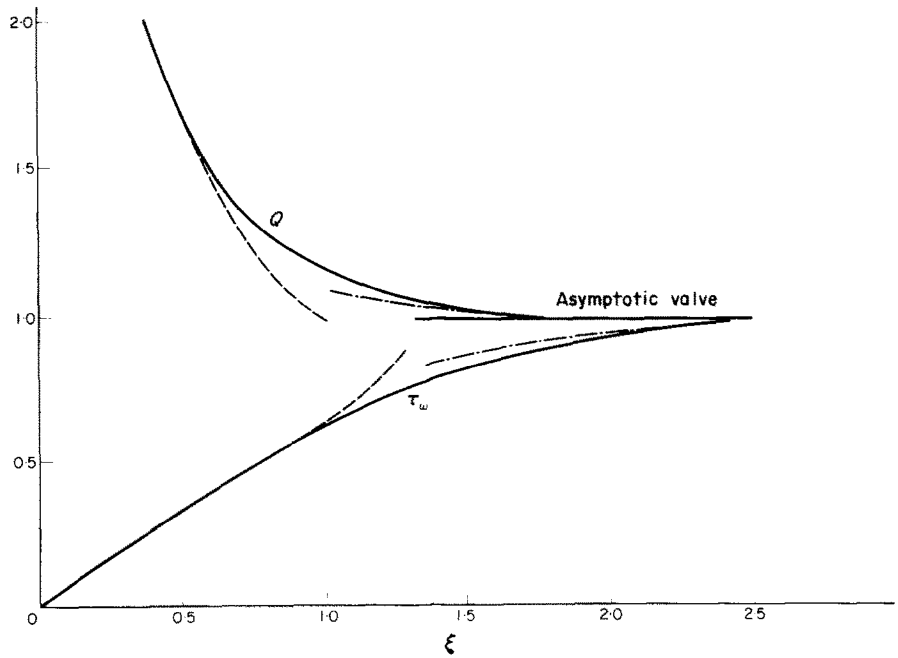

FIG. 1. Heat transfer Q and skin friction τ_ω for suction, —— numerical solution, – – – series solution of [2], –·–·– asymptotic series.

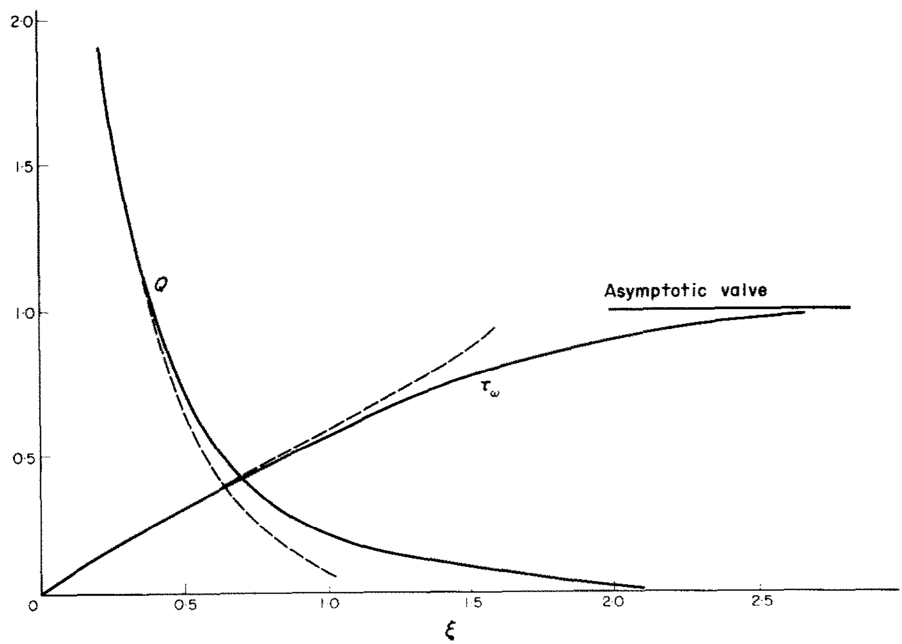

FIG. 2. Heat transfer Q and skin friction τ_ω for blowing —— numerical solution, – – – series solution of [2].

two solutions thus obtained is sufficiently small (less than 10^{-5} in the present case). The method involves inverting only simple matrices and so can be easily and quickly effected on the computer. This iterative procedure was found to converge well. The linearisation process used by Merkin [7] was found not to converge in this case.

The numerical integration started at $\xi = 0$, where the initial profiles are given by the free convection solution, and proceeded to $\xi = 3$ for suction and to $\xi = 10$ for blowing. Errors arising from using finite differences in the ξ-direction were kept small by integrating from ξ_1 to ξ_2 in first one and then two steps and ensuring that the difference between the two solutions was less than $5 \,.\, 10^{-4}$. The integration in the η-direction was started with η taking the values $\eta = 0{\cdot}1\ (0{\cdot}1)\ 10$. In the case of suction the outer value of η was reduced as ξ increased, but in the case of blowing it had to be increased to satisfy the outer boundary conditions sufficiently accurately. In this way an overall accuracy of at least three figures was achieved.

The integration was carried out for the case of $\sigma = 1$, although the computer program was written for any value of σ. Graphs of the skin friction

$$\tau_\omega = \frac{V}{g\beta\Delta T}\left(\frac{\partial u}{\partial y}\right)_{y=0} = \xi\left(\frac{\partial^2 f}{\partial \eta^2}\right)_{\eta=0}$$

and the heat transfer

$$Q = \frac{\nu}{V\Delta T}\left(\frac{\partial T}{\partial y}\right)_{y=0} = -\frac{1}{\xi}\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=0}$$

for suction and blowing are given in Figs. 1 and 2 respectively. Also presented in Figs. 1 and 2, for comparison, are Q and τ_ω obtained by using the series expansion given in [2]. It was found that this method gave, for $\sigma = 1$,

$$Q = \xi^{-1}(0{\cdot}567 \pm 0{\cdot}483\,\xi + \ldots)$$

$$\tau_\omega = 0{\cdot}642\,\xi \pm 0{\cdot}040\,\xi^2 + \ldots$$

Figures 1 and 2 show that both suction and blowing affect the heat transfer to the fluid far more readily than the skin friction.

4. ASYMPTOTIC SOLUTION—SUCTION

The fluid in the boundary layer is accelerated by the buoyancy forces caused by the applied temperature difference. Without suction the thickness of the layer increases like $x^{\frac{1}{4}}$, and the amount of heat transferred to the fluid to keep the temperature of the plate at T_1 decreases to zero like $x^{-\frac{1}{4}}$. The effect of suction is to remove the warmest fluid next to the plate, and to reduce the acceleration of the fluid in the boundary layer. This has the effect of decreasing the boundary layer thickness and increasing the amount of heat that must be supplied to the fluid to keep the plate temperature at T_1, so that asymptotically (as $x \to \infty$) the thickness of the boundary layer approaches a constant value (i.e. velocity and temperature profiles become independent of x). Because of this solution, it is not possible to obtain an asymptotic expansion for the velocity and temperature in inverse powers of x. The boundary layer approaches the asymptotic form through a term of $0(e^{-\lambda X})$, where $X = \xi^4$ and λ is the smallest eigenvalue of a set of homogeneous ordinary differential equations with homogeneous boundary conditions. A similar situation was encountered by Stewartson [5] in considering the analogous problem of the flow of a uniform stream over a plate with constant suction. He showed that, for large x, the next approximation to the 'asymptotic suction profile' was an exponentially small term.

Introduce non-dimensional variables by writing

$$\psi = \nu^2 \frac{g\beta\Delta T}{V^3} F(X, Y)$$

$$T - T_0 = \Delta T\, \theta(X, Y)$$

where

$$Y = \frac{Vy}{\nu} \equiv \eta\xi \quad \text{and} \quad X = \frac{4V^4 x}{\nu^2 g\beta\Delta T} \equiv \xi^4.$$

Equations (1)–(3) become

$$\frac{\partial^3 F}{\partial Y^3} + \theta = 4\left(\frac{\partial F}{\partial Y}\frac{\partial^2 F}{\partial X \partial Y} - \frac{\partial F}{\partial X}\frac{\partial^2 F}{\partial Y^2}\right) \quad (8)$$

$$\frac{1}{\sigma}\frac{\partial^2 \theta}{\partial Y^2} = 4\left(\frac{\partial F}{\partial Y}\frac{\partial \theta}{\partial X} - \frac{\partial F}{\partial X}\frac{\partial \theta}{\partial Y}\right) \quad (9)$$

and the boundary conditions are

$$\frac{\partial F}{\partial X} = \frac{1}{4}, \quad \frac{\partial F}{\partial Y} = 0, \quad \theta = 1 \quad \text{on} \quad Y = 0$$

and

$$\frac{\partial F}{\partial Y} \to 0, \quad \theta \to 0 \quad \text{as} \quad Y \to \infty.$$

The above discussion suggests an expansion of F and θ in the form

$$F(X, Y) = \frac{X}{4} + F_0(Y) + F_1(X, Y) \quad (10)$$

$$\theta(X, Y) = \theta_0(Y) + \theta_1(X, Y) \quad (11)$$

where, for large X, F_1 and θ_1 will be small compared with F_0 and θ_0. Putting (10) and (11) in equations (8) and (9) and equating the highest order terms leads to the equations

$$\theta_0'' + \sigma\theta_0' = 0$$

$$F_0''' + F_0'' + \theta_0 = 0$$

(dashes denote differentiation w.r.t. Y). The solution of these equations satisfying the required boundary condition is

$$\theta_0(Y) = e^{-\sigma Y} \quad (12)$$

$$F_0(Y) = \frac{(\sigma - 1) + e^{-\sigma Y} - \sigma e^{-Y}}{\sigma^2(\sigma - 1)} \quad (\text{for } \sigma \neq 1) \quad (13)$$

$$= 1 - (1 + Y)e^{-Y} \quad (\text{for } \sigma = 1).$$

The equations for the next approximation F_1 and θ_1 are found by putting (10) and (11) in equations (8) and (9) and using the fact that F_1 and θ_1 are small compared to F_0 and θ_0 to neglect products of terms in F_1 and θ_1. This leads to the equations

$$\frac{\partial^3 F_1}{\partial Y^3} + \frac{\partial^2 F_1}{\partial Y^2} + \theta_1 - 4\left(F_0'\frac{\partial^2 F_1}{\partial X \partial Y} - F_0''\frac{\partial F_1}{\partial X}\right) = 0 \quad (14)$$

$$\frac{1}{\sigma}\frac{\partial^2 \theta_1}{\partial Y^2} + \frac{\partial \theta_1}{\partial Y} - 4\left(F_0'\frac{\partial \theta_1}{\partial X} - \theta_0'\frac{\partial F_1}{\partial X}\right) = 0 \quad (15)$$

with boundary conditions

$$F_1 = \frac{\partial F_1}{\partial Y} = \theta_1 = 0 \quad \text{on} \quad Y = 0$$

$$\frac{\partial F_1}{\partial Y} \to 0, \quad \theta_1 \to 0 \quad \text{as} \quad Y \to \infty.$$

It should be noted that it is not possible to obtain a solution of equations (14) and (15) that will satisfy the boundary condition by expanding F_1 and θ_1 in any inverse power of X, and so we must look for a solution of the form

$$F_1(X, Y) = e^{-\lambda X}\phi(Y)$$

$$\theta_1(X, Y) = e^{-\lambda X}h(Y).$$

The equations for $h(Y)$ and $\phi(Y)$ are

$$\frac{1}{\sigma}h'' + h' + 4\lambda\left[\frac{(e^{-Y} - e^{-\sigma Y})}{\sigma(\sigma - 1)}h + \sigma e^{-\sigma Y}\phi\right] = 0 \quad (16)$$

$$\phi''' + \phi'' + h + \frac{4\lambda}{\sigma(\sigma - 1)} \times [(e^{-Y} - e^{-\sigma Y})\phi' + (e^{-Y} - \sigma e^{-\sigma Y})\phi] = 0. \quad (17)$$

Equations (16) and (17) hold provided $\sigma \neq 1$. When $\sigma = 1$ the equations are

$$h'' + h' + 4\lambda e^{-Y}(Yh + \phi) = 0 \quad (18)$$

$$\phi''' + \phi'' + h + 4\lambda e^{-Y}[Y\phi' + (Y - 1)\phi] = 0. \quad (19)$$

These can be found by letting $\sigma \to 1$ in (16) and (17). The boundary conditions are

$$\phi(0) = \phi'(0) = h(0) = \phi'(\infty) = h(\infty) = 0. \quad (20)$$

Equations (16) and (17) [and (18) and (19)] are homogeneous linear differential equations, and so a non-trivial solution satisfying the homogeneous boundary conditions (20) is possible only for certain values of λ (eigenvalues). No analytic solution of the equations was found and so the eigenvalues had to be found numerically in the following way. Construct two solutions (h_α, ϕ_α) and (h_β, ϕ_β) where

$$h'_\alpha(0) = 0, \qquad \phi''_\alpha(0) = 1$$

$$h'_\beta(0) = 1, \qquad \phi''_\beta(0) = 0.$$

The general solution satisfying the boundary conditions on $Y = 0$ will be of the form

$$h = \alpha h_\alpha + \beta h_\beta, \qquad \phi = \alpha\phi_\alpha + \beta\phi_\beta$$

(α and β are arbitrary constants). In general, as $Y \to \infty$

$$h \to A_i, \quad \phi'_i \sim -A_i Y + B_i \quad (i = \alpha, \beta)$$

where A_i and B_i are constants. To make h and $\phi' \to 0$ as $Y \to \infty$ we must choose α and β so that

$$\alpha A_\alpha + \beta A_\beta = 0 \quad \text{and} \quad \alpha B_\alpha + \beta B_\beta = 0.$$

A non-trivial solution of these two equations is possible only if

$$\Delta(\lambda) = A_\alpha B_\beta - A_\beta B_\alpha = 0.$$

By evaluating $\Delta(\lambda)$ for various λ, the eigenvalues can be found. The first two eigenvalues λ_1 and λ_2 are, for $\sigma = 1$,

$$\lambda_1 = 0{\cdot}1149, \qquad \lambda_2 = 0{\cdot}7127.$$

The corresponding eigenfunctions (h_1, ϕ_1) and (h_2, ϕ_2) can be determined only to within an arbitrary multiple and taking $\phi''_i(0) = 1$, $(i = 1, 2)$, it was found that

$$h'_1(0) = 0{\cdot}2454, \qquad h'_2(0) = 2{\cdot}3500.$$

The asymptotic values of the skin friction τ_ω and heat transfer Q (as defined in section 3) are, for $\sigma = 1$,

$$\tau_\omega = 1 + \gamma\, e^{-\lambda_1 X} + \ldots$$

$$Q = 1 - 0{\cdot}2454\,\gamma\, e^{-\lambda_1 X} + \ldots$$

where γ is an arbitrary constant which cannot be found from the asymptotic solution. A comparison with the numerical solution suggests that $\gamma = -0{\cdot}24 \pm 0{\cdot}02$.

5. ASYMPTOTIC EXPANSION—BLOWING

The numerical solution shows that, for large ξ, there is a region next to the plate in which

$$q = \frac{\eta}{\xi} \quad \text{and} \quad \theta = 1, \tag{21}$$

and that the thickness of this region is proportional to ξ. In the (X, Y) co-ordinate system, this result is that

$$F = -\frac{X}{4} + \frac{Y^2}{2}, \qquad \theta = 1 \tag{22}$$

and that the thickness of this region is proportional to $X^{\frac{1}{2}}$. This result is consistent with the idea that, for strong blowing (which, in this case, corresponds to ξ being large) there is a region next to the plate made up of fluid that has been blown through the plate and in which viscosity can be neglected. The streamlines of this inviscid flow, given by (22), are the parabolae

$$Y = \left(\frac{X - X_0}{2}\right)^2.$$

$X = X_0$ being the point on the plate where the streamline emerged. Since, for $X < 0$, the fluid is at rest and at the ambient temperature, this inviscid solution must be confirmed within the region $0 \leqslant Y \leqslant (X/2)^{\frac{1}{2}}$, and must have $F = 0$, $\theta = 0$ for $Y > (X/2)^{\frac{1}{2}}$. The 'dividing streamline' $Y = (X/2)^{\frac{1}{2}}$ is the one that emerged from the leading edge. Aroesty and Cole [6] found this form for the 'dividing streamline' by neglecting the viscous terms in the Von Mises form of the boundary layer equations. Since the inviscid solution gives a discontinuity in velocity and temperature on $Y = (X/2)^{\frac{1}{2}}$, there will be a region round $Y = (X/2)^{\frac{1}{2}}$ where viscous effects have to be included.

In the asymptotic solution which is now developed, results are given only for the case when $\sigma = 1$. This much simplifies the analysis without detracting from the form of the result for a general σ.

Since $\partial F/\partial Y$ is $O(X^{\frac{1}{4}})$ and θ is $O(1)$ near the 'dividing streamline', a consideration of the orders of magnitude of the terms in equations (8) and (9) shows that the thickness of the viscous region is $O(X^{\frac{1}{4}})$. It is then more convenient to use the transformed equations (4) and (5) and to work in terms of $\phi = f - (\xi/4)$. In the transformed variables the 'dividing streamline' is $\eta = \xi/\sqrt{2}$, and the viscous region has constant thickness. This suggests putting $\zeta = \eta - (\xi/\sqrt{2})$, so that $\zeta = 0$ is the equation of the 'dividing streamline'. The inviscid inner solution is

$$\phi = \frac{\zeta}{\sqrt{2}} + \frac{\zeta^2}{2\xi}, \qquad \theta = 1 \tag{23}$$

and viscous effects are important in a region of constant thickness centred round $\zeta = 0$.

In terms of (ξ, ζ) and ϕ, equations (4) and (5) become, for $\sigma = 1$,

$$\frac{\partial^3\phi}{\partial\zeta^3} + \theta + 3\phi\frac{\partial^2\phi}{\partial\zeta^2} - 2\left(\frac{\partial\phi}{\partial\zeta}\right)^2 = \xi\left(\frac{\partial\phi}{\partial\zeta}\frac{\partial^2\phi}{\partial\zeta\partial\xi} - \frac{\partial\phi}{\partial\xi}\frac{\partial^2\phi}{\partial\zeta^2}\right) \tag{24}$$

$$\frac{\partial^2\theta}{\partial\zeta^2} + 3\phi\frac{\partial\theta}{\partial\zeta} = \xi\left(\frac{\partial\theta}{\partial\xi}\frac{\partial\phi}{\partial\zeta} - \frac{\partial\theta}{\partial\zeta}\frac{\partial\phi}{\partial\xi}\right) \tag{25}$$

with boundary conditions

$$\theta \to 0, \qquad \frac{\partial\phi}{\partial\zeta} \to 0 \quad \text{as} \quad \zeta \to \infty$$

and that the solution must merge with the inviscid solution (23) near the plate.

The form of the inviscid inner solution (23) suggests looking for a solution in the form

$$\theta(\xi,\zeta) = \theta_0(\zeta) + \frac{1}{\xi}\theta_1(\zeta) + \frac{1}{\xi^2}\theta_2(\zeta) + \ldots \tag{26}$$

$$\phi(\xi,\zeta) = \phi_0(\zeta) + \frac{1}{\xi}\phi_1(\zeta) + \frac{1}{\xi^2}\phi_2(\zeta) + \ldots \tag{27}$$

where

$$\phi_n' \to 0, \quad \theta_n \to 0 \quad \text{as} \quad \zeta \to \infty \quad (\text{n} = 0,1,2,\ldots) \tag{28}$$

(dashes denote differentiation with respect to ζ). The inner boundary condition is applied on $\zeta = -\xi/\sqrt{2}$, but, since we are looking for a solution which will hold as $\xi \to \infty$, the inner boundary condition can be applied as $\zeta \to \infty$. Since we are expanding in inverse powers of ξ, this will be justifiable provided the approach to the inviscid solution is through exponentially small terms. So that we get

$$\phi_0 \sim \frac{\zeta}{\sqrt{2}}, \quad \phi_1 \sim \frac{\zeta^2}{2}, \quad \phi_n \to 0 \quad (n = 2,3,\ldots) \tag{29}$$

$$\theta_0 \to 1, \quad \phi_n \to 0 \quad (n = 1,2,\ldots) \quad \text{as} \quad \zeta \to -\infty. \tag{29}$$

The boundary conditions (28) and (29) are now compatible with the expansion of θ and ϕ in (26) and (27).

The equations for θ_0 and ϕ_0 are

$$\theta_0'' + 3\phi_0\theta_0' = 0 \tag{30}$$

$$\phi_0''' + \theta_0 + 3\phi_0\phi_0'' - 2\phi_0'^2 = 0. \tag{31}$$

A solution of (30) and (31) satisfying the required boundary conditions was obtained by first writing them in finite difference form and then solving the resulting non-linear algebraic equations iteratively by Newton's method. Values of ϕ_0' and θ_0 are given in Table 1.

To consider the behaviour of ϕ_0 and θ_0 as $\zeta \to -\infty$, put $\theta_0 = 1 + h_0$ and $\phi_0 = (\zeta/\sqrt{2}) + g_0$, where h_0 and g_0 are small as $\zeta \to -\infty$. The approximate equations for h_0 and g_0 are

$$h_0'' + \frac{3\zeta}{\sqrt{2}}h_0' = 0 \tag{32}$$

$$g_0''' + \frac{3\zeta}{\sqrt{2}}g_0'' - \frac{4}{\sqrt{2}}g_0' = -h_0. \tag{33}$$

Table 1. Velocity and temperature functions for the asymptotic expansion (blowing)

ζ	ϕ_0'	θ_0	ϕ_1'	θ_1	ϕ_2'	θ_2	$\bar{\phi}_3'$	$\bar{\theta}_3$
−3·0	0·707	1·000	−3·000	0·000	0·000	0·000	0·001	−0·001
−2·0	0·707	0·999	−2·003	−0·004	−0·008	−0·007	−0·014	−0·006
−1·5	0·703	0·990	−1·515	−0·015	−0·027	−0·008	−0·033	−0·001
−1·0	0·686	0·950	−1·050	−0·019	−0·054	0·011	−0·051	0·001
−0·5	0·634	0·838	−0·628	0·024	−0·071	0·043	−0·058	−0·005
0·0	0·532	0·643	−0·285	0·106	−0·055	0·077	−0·050	0·000
0·5	0·396	0·420	−0·059	0·154	−0·009	0·102	−0·025	0·025
1·0	0·263	0·238	0·047	0·139	0·032	0·100	0·004	0·045
1·5	0·160	0·122	0·072	0·096	0·048	0·076	0·023	0·045
2·0	0·090	0·058	0·061	0·056	0·045	0·048	0·027	0·033
3·0	0·025	0·012	0·026	0·015	0·023	0·014	0·016	0·011
5·0	0·002	0·000	0·002	0·001	0·003	0·001	0·002	0·001

The solution of (32) is

$$h_0(\zeta) = A_0 \int_{-\infty}^{\zeta} \exp\left(-\frac{3S^2}{2\sqrt{2}}\right) dS$$

$$\sim -\frac{A_0\sqrt{2}}{3\zeta} \exp -\left(\frac{3\zeta^2}{2\sqrt{2}}\right)\left(1 - \frac{\sqrt{2}}{3\zeta^2} + \ldots\right).$$

The solution of (33) is

$$g_0(\zeta) = G_0(\zeta) + \frac{\sqrt{2}}{4}\left(\zeta h_0 + \frac{\sqrt{2}}{3} h_0'\right)$$

where G_0 is the complementary function of equation (33) such that $G_0 \to 0$ as $\zeta \to -\infty$.

$$G_0(\zeta) = B_0 \zeta \exp -\left(\frac{3\zeta^2}{2\sqrt{2}}\right) U\left(\frac{13}{6}; \frac{3}{2}; \frac{3\zeta^2}{2\sqrt{2}}\right)$$

where U is the form of the confluent hypergeometric function not exponentially large at infinity. Using the result given in [8], page 60, it follows that

$$G_0(\zeta) \sim -B_0 \left(\frac{2\sqrt{2}}{3}\right)^{13/6} \frac{e^{-3\zeta^2/2\sqrt{2}}}{|\zeta|^{10/3}} \times \left(1 - \frac{65\sqrt{2}}{27\zeta^2} + \ldots\right).$$

The equations for ϕ_1 and θ_1 are

$$\theta_1'' + 3\phi_0\theta_1' + 2\phi_1\theta_0' + \phi_0'\theta_1 = 0 \quad (34)$$

$$\phi_1''' + \theta_1 + 3\phi_0\phi_1'' - 3\phi_0'\phi_1' + 2\phi_0''\phi_1 = 0 \quad (35)$$

with

$$\theta_1 \to 0, \quad \phi_1' \to 0 \quad \text{as} \quad \zeta \to \infty$$

$$\theta_1 \to 0, \quad \phi_1 \sim \frac{\zeta^2}{2} \quad \text{as} \quad \zeta \to -\infty. \quad (36)$$

In general, any solutions of equations (34) and (35) will have

$$\theta_1 \to \alpha_1, \quad \phi_1 \sim -\frac{\alpha_1\zeta}{3C} + \beta_1 \quad \text{as} \quad \zeta \to \infty$$

and

$$\theta_1 \sim \gamma_1|\zeta|^{-\frac{4}{3}}, \quad \phi_1 \sim \delta_1\zeta^2 + \varepsilon_1 - \frac{3\sqrt{2}}{8}\gamma_1|\zeta|^{\frac{2}{3}}$$

as $\zeta \to \infty$; $C = \lim_{\zeta\to\infty} \phi_0(\zeta) = 0{\cdot}544045$. The solution which satisfies (36) must have $\alpha_1 = \beta_1 = \gamma_1 = \epsilon_1 = 0$, $\delta_1 = \frac{1}{2}$ to make the approach to the inviscid solution through exponentially small terms. Equations (34) and (35) were solved numerically in the following way. Five numerical integrations of the equations were performed, giving (f_i, t_i) $(i = 1, 2, \ldots 5)$, in each case starting from $\zeta = 0$ where one of the initial quantities $\phi_1(0)$, $\phi_1'(0)$, $\phi_1''(0)$, $\theta_1(0)$, $\theta_1'(0)$ needed to perform the integration was taken as 1 and all the others as 0. The integration proceeded first to large positive ζ and the constants α_{1i} and β_{1i} were determined. The equations were then integrated up to large negative ζ and the constants γ_{1i}, δ_{1i} and ϵ_{1i} determined. The general solution of (34) and (35) is

$$\theta_1 = \sum_{i=1}^{5} A_i t_i(\zeta) \qquad \phi_1 = \sum_{i=1}^{5} A_i f_i(\zeta)$$

where the A_i are constants. In order to satisfy (36), the A_i must be chosen so that

$$\sum_{i=1}^{5} A_i\alpha_{1i} = \sum_{i=1}^{5} A_i\beta_{1i} = \sum_{i=1}^{5} A_i\gamma_{1i} = \sum_{i=1}^{5} A_i\epsilon_{1i} = 0$$

$$\sum_{i=1}^{5} A_i\delta_{1i} = \tfrac{1}{2}. \tag{37}$$

Having solved (37) for the A_i, the correct initial values can be found and the equations integrated numerically to satisfy (36). Values of ϕ_1' and θ_1 are given in Table 1.

To consider the behaviour of ϕ_1 and θ_1 as $\zeta \to -\infty$, put $\phi_1 = (\zeta^2/2) + g_1$ (g_1 small as $\zeta \to -\infty$) in (34) and (35). The solutions of the resulting equations which hold as $\zeta \to -\infty$ are found to be

$$\theta_1 = h_1(\zeta) + \left(\frac{\zeta^2 + \sqrt{2}}{4\sqrt{2}}\right) h_0'(\zeta)$$

where

$$h_1(\zeta) = A_1 \exp\left(-\frac{3\zeta^2}{2\sqrt{2}}\right) U\left(\frac{1}{3};\frac{1}{2};\frac{3\zeta^2}{2\sqrt{2}}\right)$$

$$\sim A_1 \left(\frac{2\sqrt{2}}{3}\right)^{\frac{1}{3}} |\zeta|^{-\frac{2}{3}} \exp\left(-\frac{3\zeta^2}{2\sqrt{2}}\right) \times \left(1 - \frac{5\sqrt{2}}{27\zeta^2} + \ldots\right)$$

and

$$g_1(\zeta) = G_1(\zeta) + \frac{\sqrt{2}}{8}(3\zeta h_1 + \sqrt{2}h_1') + \frac{\sqrt{2}}{48}(8h_0 - \zeta h_0') + \frac{\sqrt{2}}{40} \times [(5\zeta^2 + 12\sqrt{2})G_0' - 9\zeta G_0]$$

where

$$G_1(\zeta) = B_1\zeta \exp\left(-\frac{3\zeta^2}{2\sqrt{2}}\right) U\left(2;\frac{3}{2};\frac{3\zeta^2}{2\sqrt{2}}\right)$$

$$\sim -\tfrac{8}{9}B_1 \frac{\exp(-3\zeta^2/2\sqrt{2})}{|\zeta|^3}\left(1 - \frac{2\sqrt{2}}{\zeta^2} + \ldots\right).$$

The equations for θ_2 and ϕ_2 are

$$\theta_2'' + 3\phi_0\theta_2' + 2\phi_0'\theta_2 + \phi_2\theta_0' = -2\phi_1\theta_1' - \phi_1'\theta_1 \tag{38}$$

$$\phi_2''' + \theta_2 + 3\phi_0\phi_2'' - 2\phi_0'\phi_2' + \phi_0''\phi_2 = \phi_1'^2 - 2\phi_1\phi_1'' \tag{39}$$

with

$$\theta_2 \to 0, \quad \phi_2' \to 0 \quad \text{as} \quad \zeta \to \infty$$
$$\theta_2 \to 0, \quad \phi_2 \to 0 \quad \text{as} \quad \zeta \to -\infty. \tag{40}$$

Equations (38) and (39) were solved numerically in a way similar to the numerical solution of (34) and (35). As in that case, the 5 complementary functions were found first, then a particular integral was found by integrating the full equations. These 6 solutions were then combined in such a way as to satisfy (40) and to make the approach to the inviscid solution through exponentially small terms. Values of ϕ_2' and θ_2 are given in Table 1. The behaviour of ϕ_2 and θ_2 as $\zeta \to -\infty$ has been treated in a way similar to ϕ_1 and θ_1. The results are long and not of sufficient interest to quote here.

The equations for θ_3 and ϕ_3 are

$$\theta_3' + 3\phi_0\theta_3 = -2\phi_1\theta_2 - \phi_2\theta_1 \tag{41}$$

$$\phi_3''' + \theta_3 + 3\phi_0\phi_3'' - \phi_0'\phi_3' = \phi_1'\phi_2' - \phi_2\phi_1'' - 2\phi_1\phi_2'' \tag{42}$$

with

$$\theta_3 \to 0, \quad \phi_3' \to 0 \quad \text{as} \quad \zeta \to \infty$$
$$\theta_3 \to 0, \quad \phi_3 \to 0 \quad \text{as} \quad \zeta \to -\infty. \tag{43}$$

The equation obtained directly from (25) has been integrated once to give (41).

Equations (41) and (42) possess a complementary function (F_3, H_3), given by

$$F_3 = \phi_0' - \frac{1}{\sqrt{2}}, \qquad H_3 = \theta_0', \tag{44}$$

which satisfies (43). As any multiple of (F_3, H_3) added a solution of (41) and (42) satisfying (43) will still satisfy (41)–(43), it follows that the solution for θ_3 and ϕ_3 can be determined only to within an arbitrary multiple of (F_3, H_3).

Equations (41) and (42) were integrated numerically for the case when $\theta_3(0) = 0$, and the values of θ_3 and ϕ'_3 are given in Table 1. If we call this solution $(\bar{\phi}_3, \bar{\theta}_3)$, then the full solution will be

$$\theta_3 = \bar{\theta}_3 + \lambda H_3, \quad \phi_3 = \bar{\phi}_3 + \lambda F_3.$$

λ is an arbitrary constant which cannot be found from the asymptotic solution. This arbitrariness is due to the boundary condition at $x = 0$ not being taken into account in the asymptotic expansion. It is interesting to note that a solution of (41) and (42) has been obtained which satisfies all the boundary conditions imposed without the inclusion of a logarithmic term in the expansion, as seems to be the usual case when a complementary function like (44) is encountered, see [5].

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CONVECTION NATURELLE AVEC SOUFFLAGE ET ASPIRATION

Résumé—On considère les effets de soufflage uniforme et de succion sur la couche limite de convection naturelle sur une plaque verticale. On obtient dans chacun des cas une solution numérique des équations de la couche limite. Dans le cas de la succion, on trouve que la solution asymptotique est une couche limite d'épaisseur constante. L'approche de cette solution est discutée. Quand le fluide est soufflé à travers la plaque on trouve que, à de grandes distances du bord d'attaque, la couche limite a une région interne sans viscosité de fluide soufflé à travers la plaque et une région externe visqueuse où le fluide atteint les conditions ambiantes.

FREIE KONVEKTION MIT EINBLASUNG UND ABSAUGUNG

Zusammenfassung—Der Einfluss gleichmässiger Einblasung und Absaugung auf die Grenzschicht bei freier Konvektion an einer vertikalen Platte wird behandelt. Eine numerische Lösung der vollständigen Grenzschicht-Gleichungen wurde in beiden Fällen erreicht. Im Falle der Absaugung ergab sich eine Grenzschicht mit konstanter Dicke als asymptotische Lösung. Der Lösungsweg wird ebenfalls diskutiert. Wird ein Fluid durch die Platte geblasen, so findet man, dass in weitem Abstand von der Anlaufkante die Grenzschicht einen inneren, reibungsfreien Bereich aufweist, gebildet von dem eingeblasenen Fluid, und einen äusseren Bereich, wo das Fluid Umgebungsbedingungen annimmt.

СВОБОДНАЯ КОНВЕКЦИЯ ПРИ ВДУВЕ И ОТСОСЕ

Аннотация—Рассматриваются влияния равномерного вдува и отсоса на пограничный слой на вертикальной пластине при свободной конвекции. В обоих случаях получено численное решение уравнений полностью развитого пограничного слоя. Найдено, что при отсосе асимптотическое решение описывает пограничный слой постоянной толщины. Также обсуждается подход к этому решению. При вдуве жидкости через пластину найдено, что на больших расстояниях от передней кромки пограничный слой имеет внутреннюю область невязкой жидкости, вдуваемой через пластину, и внешнюю область вязкой жидкости, достигающей условий окружающей среды.